

Flows on a separable C^* -algebra

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Motivations

- Unbounded derivations
(as a continuation of the theory of derivations)
- Applications to Physics
(KMS states, ground states, --)
- Actions of $\mathbb{R}$
(instead of locally compact abelian groups)

Why flows ?

easy to construct
easy to perturb

References

- G. K. Pedersen "C*-algebras and their automorphism groups" 1979
- O. Bratteli, D. W. Robinson
"Operator Algebras and Quantum Statistical Mechanics" I, II, 1979, 1981
- S. Sakai
"Operator Algebras in Dynamical Systems" 1991

I mostly study flows in the spirit of Pedersen's book.

Perturbations

Let α be a flow on A ,

$$\left(\begin{array}{l} t \in \mathbb{R} \rightarrow \alpha_t \in \text{Aut}(A) \quad \text{homomorphism} \\ t \mapsto \alpha_t(x) \quad \text{cont. for } x \in A \end{array} \right)$$

Let δ_α be the generator of α : $\alpha_t = e^{t\delta_\alpha}$

- For $h = h^* \in A$, $\alpha^{(h)}$ denotes the flow generated by $\delta_\alpha + \text{ad } ih$, called an inner perturbation of α .

- Let u be an α -cocycle

$$u : \mathbb{R} \rightarrow \mathcal{U}(A) \text{ (or } \mathcal{U}(M(A)) \text{)}$$

$$u_s \alpha_s(u_t) = u_{s+t}$$

Then $t \mapsto \text{Ad } u_t \alpha_t$ is a flow,

called a cocycle perturbation of α

Suppose $A \ni 1$.

If the α -cocycle u is differentiable and $i \cdot h = \frac{d u_t}{d t} \Big|_{t=0}$, then

$$\text{Ad } u \alpha = \alpha^{(h)}$$

In general the α -cocycle u is given as

$$u_t = w v_t \alpha_t(w^*)$$

where $w \in \mathcal{U}(A)$

v : differentiable α -cocycle

Hence $\text{Ad } u \alpha$ is of the form

$$\text{Ad } w \circ \alpha^{(h)} \circ \text{Ad } w^*$$

Hence in the unital case
there is not much difference
between inner perturbations
and cocycle perturbations.

But in the non-unital case
I do not know.

(In this case $t \mapsto u_t$ is cont.
in the strict topology.)

- In general we want to study
properties which are invariant
under cocycle perturbations.

Properties for flows

- AI (= approximately inner)
- Rohlin
- Spectra
e.g. Connes spectrum
- KMS states (as a bundle over
the set of temperatures)
-

Remarks

- I do not know if AI is invariant under cocycle perturbations
- KMS states should be replaced by KMS positive functionals (to make it invariant under cocycle perturbations)

Covariant Representations

$$(A, \alpha)$$

$$(\pi, U) \quad U_t \pi(x) U_t^* = \pi \alpha_t(x), \quad x \in A$$

Problems :

Are there many covariant representations?

What is the Connes spectrum for $\bar{\alpha}$:

$$\bar{\alpha}_t = \text{Ad } U_t \quad \text{on } \pi(A)'' \quad ?$$

--- This can be answered by

adapting Glimm's theorem for antiliminary C^* -algebras.

Theorem

A : separable prime (antiliminary)

α : flow on A , the Connes spectrum $R(\alpha) \neq 0$.

Then the following are equivalent:

(1) $\exists$ faithful family of cov. irreducible rep.

(2) $\exists$ faithful cov. irreducible rep (π, U)
 $\ker(\pi \times U)$: invariant under $\hat{\alpha}|_{R(\alpha)}$

(3) $\forall D = \bigotimes_{n=1}^{\infty} M_{k_n} \quad \forall \varepsilon > 0$
 $\forall \gamma_t = \bigotimes Ad e^{it h_n} \quad Sp(\gamma) \subset R(\alpha)$

$\exists B \subset A, h \in A_{sa}, \eta \in A^{**}$ closed

$$\|h\| < \varepsilon$$

$$\alpha_t^{(h)}(B) = B \quad \alpha_t^{(h)}(\eta) = \eta$$

$$\eta A \eta = B \eta$$

$$(B \eta, \alpha^{(h)}|_{B \eta}) \cong (D, \gamma)$$

$$\forall x \in A \quad x c(\eta) = 0 \Rightarrow x = 0$$

(where $c(\eta)$ the central support of η
in A^{**})

Note : For compact group actions,
a similar result is obtained
without perturbations (Bratteli-Robinson-K)

Problem Can we generalize this
to actions of $\mathbb{R}^2$?

So the problem on covariant representations reduces to the one on the UHF flow γ , which is easy.

Note in the above situation

$$\forall \varepsilon > 0, \quad \exists h = h^* \in A, \quad \|h\| < \varepsilon$$

the set of $\alpha^{(h)}$ -invariant states include the state space of a UHF algebra.

Note the condition

$\exists$ faithful family of cov. irreducible representations

seems to follow or is required below.

- $\forall \pi_1, \pi_2$: irreducible rep of separable
 A s.t. $\text{Ker } \pi_1 = \text{Ker } \pi_2$

$\exists \gamma$: (asymptotically inner) automorphism
 of A

$$\pi_1 \gamma \sim \pi_2. \quad (\text{due to Ozawa, Sakai, K})$$

Problem Can we generalize it to
 (A, α)

Symmetry group for α

$$G_\alpha = \left\{ \gamma \in \text{Aut } A \mid \gamma \alpha \gamma^{-1} : \text{cyclic perturb. of } \alpha \right\}$$

Theorem

A separable prime C^* -alg

α flow $R(\alpha) \neq 0$

$\forall (\pi_1, U_1), (\pi_2, U_2)$ cov. irr. rep

$$\text{Ker } \pi_1 \times U_1 = \text{Ker } \pi_2 \times U_2$$

$$\Rightarrow \gamma \in \text{Aut}(A)$$

$\gamma \alpha \gamma^{-1}$: cocycle perturbation of α

$$\pi_1 \gamma \sim \pi_2.$$

Remark The above γ extends to
an asymptotically inner aut. $\bar{\gamma}$ of $A \rtimes \mathbb{R}$
s.t. $(\pi_1 \times U_1) \bar{\gamma} \sim \pi_2 \times U_2.$

Remark : If π is the GNS rep. associated
with a KMS state, then $\pi \gamma \simeq \pi$

[Fannes, Vanheuverzwijn, Verbeune]

AI (= approximately inner)

There is no good characterization except for the definition.

- If A is unital and has a tracial state, then

$\exists$ KMS states for all temperatures.

- (π, U) cov. rep on a separable H_π
 $\pi(A)''$ type I
 $\exists (h_n) \quad \text{Ad } e^{it h_n}(x) \rightarrow \alpha_t(x), x \in A$
 $\pi(e^{it h_n}) \rightarrow U_t$ strongly

Theorem

A : separable antiliminary

$\exists \alpha$: AI flow, $\mathbb{R}(\alpha) = \mathbb{R}$.

Remark "The Connes spectrum $\mathbb{R}(\alpha) = \mathbb{R}$ "
can be replaced by

$\forall O \subset \mathbb{R}$ non-empty open

$\exists (z_n)$ in $A^\alpha(O)$ central sequence
non-trivial in the sense
 $\lim_n \|x z_n\| = 0 \Rightarrow x = 0 \quad (\forall x \in A)$

Problem

Are there many cocycle conjugacy
classes of flows on A ?

Rohlin flows

Definition α has the Rohlin property

if $\forall p \in \mathbb{R} \quad \exists (u_n) \in \mathcal{U}(M(A))$

$$\|\alpha_t(u_n) - e^{ipt} u_n\| \rightarrow 0$$

$$\|[u_n, x]\| \rightarrow 0, \quad x \in A$$

Example

$$A_\theta = C^*(u, v), \text{ where } uv = e^{2\pi i \theta} vu$$

$$\theta \notin \mathbb{Q}$$

$$\alpha_t(u) = e^{2\pi i t} u$$

$$\alpha_t(v) = e^{2\pi i p t} v$$

$\Rightarrow$ If $p \notin \mathbb{Q}$, α has the Rohlin property.

$1, p$ are linearly independent over $\mathbb{Z} + \theta\mathbb{Z}$

$\rightsquigarrow$ generalized to simple AT algebras of KRO (thanks to Elliott's classification th.)

Lemma : Let α be a Rohlin flow on a unital C^* -algebra A and $L > 0$.

Let γ denote the flow on $C(\mathbb{R}/L\mathbb{Z})$ by translations. Then

$$\exists (\phi_n) : A \otimes C(\mathbb{R}/L\mathbb{Z}) \rightarrow A$$

approximate homomorphism

$$\phi_n(a \otimes 1) = a, \quad a \in A$$

$$\| \phi_n \circ \alpha_t \otimes \gamma_t(x) - \alpha_t \phi_n(x) \| \rightarrow 0$$

for $x \in A \otimes C(\mathbb{R}/L\mathbb{Z})$

Proof Let (u_n) in $\mathcal{U}(A)$ be such that

$$\| [u_n, a] \| \rightarrow 0$$

$$\| \alpha_t(u_n) - e^{i \frac{t}{L}} u_n \| \rightarrow 0$$

Then ϕ_n is defined by

$$\phi_n(a \otimes z) = a u_n, \quad a \in A$$

Theorem :

If α has the Rohlin property

(for a unital A), then

an α -cocycle is almost a coboundary

$l^\infty(A)$ bounded sequences in A
 $C_0(A)$ " converging to 0

$$A^\infty = l^\infty(A) / C_0(A)$$

$l_\alpha^\infty(A)$: α -continuous part of $l^\infty(A)$

$$A_\alpha^\infty = l_\alpha^\infty(A) / C_0(A)$$

More precisely

If $u : \mathbb{R} \rightarrow \mathcal{U}(A' \cap A_\alpha^\infty)$ is an α -cocycle

then $\exists v \in \mathcal{U}(A' \cap A_\alpha^\infty)$

$$u_t = v \alpha_t(v^*)$$

Prop. Let A be a separable purely infinite simple C^* -algebra and α a Rohlin flow on A . Then for any $\varepsilon > 0$

$$\exists h = h^* \in A$$

$$\exists \varphi \text{ pure state of } A$$

$$\exists (e_n) \text{ decreasing sequence of projections in } A$$

$$\alpha_t^{(h)}(e_n) = e_n$$

$$\lim e_n = \text{the support projection of } \varphi$$

Prop. Let A be a separable purely infinite simple C^* -algebra and α a Rohlin flow on A . Then $A \rtimes_{\alpha} \mathbb{R}$ is purely infinite and simple.

Proof

A simple $\Rightarrow A \rtimes_{\alpha} \mathbb{R}$ simple

because $\hat{\alpha}_p$ is approximately inner

so any ideal of $A \rtimes_{\alpha} \mathbb{R}$

is α -invariant

To show "purely infinite" we mimic the proof for the case of Cuntz algebra

$\mathcal{O}_n = C^*(s_1, \dots, s_n)$ Cuntz algebra

$$\sum_{i=1}^n s_i s_i^* = 1$$

$$s_i^* s_i = 1.$$

For $(p_1, \dots, p_n)$ in $\mathbb{R}$ define a flow α on $\mathcal{O}_n$ by

$$\alpha_t(s_i) = e^{ip_i t} s_i, \quad i=1, \dots, n.$$

Then

$\mathcal{O}_n$: separable purely infinite simple nuclear

α : non AI flow for $(p_1, \dots, p_n) \neq 0$

Theorem

Let α be the quasi-free flow on $\mathcal{O}_n$ corresponding to $p_1, p_2, \dots, p_n$. Then the following are equivalent.

- (1) α has the Rohlin property.
- (2) $\mathbb{Q}_n \rtimes_{\alpha} \mathbb{R}$ is purely infinite & simple
- (3) $p_1, \dots, p_n$ generate $\mathbb{R}$ as a closed subsemigroup.

In this case all such α are cocycle conjugate with each other.

i.e., if α and β are as above,

$$\exists \phi \in \text{Aut } \mathcal{O}_n, \quad \exists u : \alpha\text{-cocycle}$$

$$\text{Ad } u \alpha = \phi \beta \phi^{-1}$$

Proof (3) $\Rightarrow$ (1)

- For $p \in \mathbb{R}$, find $u \in \mathcal{U}(\mathcal{O}_n)$ of the form $\sum_i S_{I_i} S_{J_i}^*$, where I_i, J_i are finite sequences in $\{1, 2, \dots, n\}$ s.t.

$$\alpha_t(u) \simeq e^{ip t} u$$

- $\exists (v_k)$ in $\mathcal{U}(\mathcal{O}_n)$

$$\alpha_t(v_k) = v_k$$

$(v_k s_i)_k$: central sequence

(Thus (v_k) must satisfy :

$$v_k \lambda(v_k^*) \simeq \sum_{i,j=1}^n s_i s_j^* s_i^* s_j$$

where $\lambda(\cdot) = \sum_{i=1}^n s_i \cdot s_i^*$)

Define an endomorphism ϕ_k of $\mathcal{O}_n$ by

$$\phi_k(s_i) = v_k s_i, \quad i=1, \dots, n.$$

Then $\phi_k \alpha_t = \alpha_t \phi_k$

and for the unitary u with $\alpha_t(u) \simeq e^{itp} u$,

$\phi_k(u)$ satisfies

$$\alpha_t(\phi_k(u)) \simeq e^{itp} \phi_k(u)$$

$(\phi_k(u))$ central

Note :

$$U = \sum_{i,j=1}^n s_i s_j^* s_i^* s_j^* \in M_n \otimes M_n \subset \bigotimes_N M_n \subset \mathcal{O}_n$$

$\lambda \upharpoonright \bigotimes_N M_n$: one-sided shift

To show that there is such a (V_k)

with $\|V_k \lambda(V_k^*) - U\| \rightarrow 0$

we show

Lemma $B = \bigotimes_{\mathbb{Z}} M_n$

$$\gamma: T^{n-1} \rightarrow \text{Aut}(B)$$

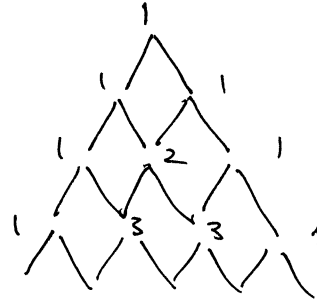
$$z \mapsto \bigotimes_{\mathbb{Z}} \text{Ad} \begin{pmatrix} 1 & z_1 & 0 \\ & \ddots & \\ 0 & & z_{n-1} \end{pmatrix}$$

$$A = B^\gamma$$

σ : shift on B

$\Rightarrow \sigma|_A$ has the one-cocycle property.

A is an AF algebra, whose Bratteli diagram looks like Pascal's triangle for $n=2$ and a higher dimensional version for a general n .



A has n characters

$\phi_1, \dots, \phi_n$.

" $\sigma|A$ has the one-cocycle property"

means

$\forall u = (u_n) \in B' \cap A^{\text{cl}}$ unitary

$$\phi_i(u) = \lim \phi_i(u_n) = 1$$

$\exists v \in U(B' \cap A^{\text{cl}})$

$$u = v \sigma(v^*)$$

The shift σ on $B = \bigotimes_{\mathbb{Z}} M_n$ has the one-cocycle property (or Rohlin property)

[Bratteli - Rørdam - Størmer - K]. Since B has a unique tracial state, some qualitative arguments are enough to conclude this, based on the result for type II_1 factors due to Connes (which yielded an "approximate Rohlin property"). Since the factorial tracial state space of A is a $(n-1)$ -dim. simplex, some qualitative arguments are required to prove "the one-cocycle property for $\sigma|_A$ ".

For $n = 2$

this is obtained from the
CAR algebra formalism (like
the BRSK arguments for $\bigotimes_{\mathbb{Z}} M_2$)

For $n > 2$ By Induction

$$\bigotimes_1^L M_n \hookrightarrow \bigotimes_{\mathbb{Z}} M_{n+1}$$

For a free ultrafilter ω on $\mathbb{N}$

$$C_\omega(A) = \{x \in \ell^\omega(A) \mid \lim_\omega \|x_n\| = 0\}$$

$$A^\omega = \ell^\omega(A) / C_\omega(A)$$

$$A_\kappa^\omega = \ell_\kappa^\omega(A) / C_\omega(A)$$

Then $A' \cap A^\omega$ purely infinite and simple
if A is unital separable nuclear &
purely infinite simple

Based on this result and classification
result due to Kirchberg and Phillips
we show :

Theorem

A : unital sep. nuc. purely inf. simple
 C^* -algebra satisfying UCT.

α : flow on A .

Then the following are equivalent:

- (1) α has the Rohlin property
- (2) $(A' \cap A_\alpha^\omega)^\alpha$ purely infinite simple
 $K_0((A' \cap A_\alpha^\omega)^\alpha) \cong K_0(A' \cap A^\omega)$
 $\text{Sp}(\alpha|_{A' \cap A_\alpha^\omega}) = \mathbb{R}$
- (3) $A \rtimes_\alpha \mathbb{R}$ purely infinite and simple
 $\hat{\alpha}$: Rohlin flow
- (4) $A \rtimes_\alpha \mathbb{R}$ purely infinite and simple
 Each α_t is α -invariantly
 approximately inner
 $\left(\begin{array}{l} \forall t \in \mathbb{R}, \exists (u_n) \text{ in } \mathcal{U}(A) \\ \alpha_t = \lim \text{Ad } u_n \\ \| \alpha_s(u_n) - u_n \| \rightarrow 0 \end{array} \right)$

In this case

$$K_1((A' \cap A_\infty^\omega)^\alpha) \cong K_1(A' \cap A^\omega)$$

also follows.

Proof

$$(1) \Rightarrow (4) \Rightarrow (3) \xRightarrow{\text{Takesaki's duality}} (1)$$

and

$$(1) \Leftrightarrow (2)$$

Problem

Is " $A \rtimes_\alpha \mathbb{R}$ purely inf & simple" enough
to conclude the Rohlin property?

$$(1) \Rightarrow (4)$$

Fix t

$$\exists (u_n) \quad \alpha_t = \lim \text{Ad } u_n$$

$$\left(s \mapsto u_n^* \alpha_s(u_n) \right) : \text{central seq of } \alpha\text{-cycles}$$

$$\exists (v_n) \in A' \cap A^\omega$$

$$\| u_n^* \alpha_s(u_n) - v_n \alpha_s(v_n^*) \| \rightarrow 0$$

Then

$$\alpha_t = \lim \text{Ad } (u_n v_n)$$

$$\| \alpha_s(u_n v_n) - u_n v_n \| \rightarrow 0$$

$$(4) \Rightarrow (3) \quad \text{If } w_n = u_n v_n \text{ as above}$$

$$(w_n^* \lambda(t)) \text{ central in } A \rtimes \mathbb{R}$$

$$\hat{\alpha}_p(w_n^* \lambda(t)) = e^{i p t} w_n^* \lambda(t)$$

Suppose that α has the Rohlin property.

Proof of $K_0((A' \cap A^\omega)^\vee) \rightarrow K_0(A' \cap A^\omega)$ surjective

$\therefore e \in A' \cap A^\omega$ proj.

$$e = (e_n)$$

$T \gg 0$ Define $E_n \in (A' \cap C[0, T] \otimes A)^\omega$ by

$$E_n(t) = \alpha_t(e_n)$$

$$\bullet \exists (U_n) \in A' \cap (C[0, T] \otimes A)^\omega$$

$$Ad U_n(t)(E_n(0)) = E_n(t)$$

$$\bullet \exists (V_n) \in A' \cap (C[0, T] \otimes A)^\omega$$

$$V_n(0) = 1$$

$$V_n(T) = U_n(T)^*$$

$$\|V_n(s_2) - V_n(s_1)\| < C|s_2 - s_1|/T$$

(due to Nakamura based on K-P's

$$O_\infty \hookrightarrow A)$$

Define $F_n \in C(\mathbb{R}/T\mathbb{Z}) \otimes A$ by

$$\begin{aligned} F_n(t) &= \alpha_{t-T}(V_n(t)) \alpha_t(e_n) \alpha_{t-T}(V_n(t)^*) \\ &= \text{Ad}(\alpha_{t-T}(V_n(t)) U_n(t))(e_n) \\ \left(\Rightarrow F_n(0) = e_n = F_n(T) \right) \end{aligned}$$

If γ is the flow on $C(\mathbb{R}/T\mathbb{Z})$ by translations, then

$$\|\gamma_t \otimes \alpha_t(F_n) - F_n\| \leq 2C|t|/T.$$

By using the approximate homomorphism (ϕ_k) of $C(\mathbb{R}/T\mathbb{Z}) \otimes A$ into A

with $\phi_k \circ \gamma_t \otimes \alpha_t \simeq \alpha_t \circ \phi_k$, we

define $f = (\phi_{k_n}(F_n))$, which belongs

to $(A' \cap A_\alpha^\omega)^\times$ and satisfies

$$f \sim e.$$

Proof of $K_0((A' \cap A^\omega)^\alpha) \rightarrow K_0(A' \cap A^\omega)$ injective

$$e, f \in (A' \cap A^\omega)^\alpha \quad \text{proj.}$$

$$e \sim f \quad \text{in } A' \cap A^\omega$$

We may assume that

$$\begin{aligned} e &= (e_n) & e_n & \text{proj.} \\ f &= (f_n) & f_n & \text{proj.} \end{aligned}$$

$$\delta_\alpha(e_n) = 0$$

$$\delta_\alpha(f_n) = 0$$

$$\exists u = (u_n) \in A' \cap A^\omega$$

$$u_n^* u_n = e_n$$

$$u_n u_n^* = f_n$$

$$\text{Let } z_n(t) = u_n^* \alpha_t(u_n) \in e_n A e_n$$

$$z_n : \alpha|_{e_n A e_n} \text{- cycle}$$

$$\exists y_n \in \mathcal{U}(e_n A e_n)$$

$$\| z_n(t) - y_n \alpha_t(y_n^*) \| \rightarrow 0$$

$$(y_n) \in A' \cap A^\omega$$

Then $(u_n y_n) \in (A' \cap A_\alpha^u)^\alpha$

and

$$(u_n y_n)^\star (u_n y_n) = e_n$$

$$(u_n y_n) (u_n y_n)^\star = f_n$$

i.e., $e. = (e_n) \sim f. = (f_n)$ in $(A' \cap A_\alpha^u)^\alpha$