

Recent Challenges in Multifractal Analysis

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Motivation

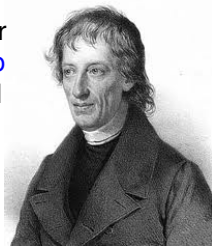
Purpose of multifractal analysis : Introduce and study classification and model selection parameters for data (functions, measures, distributions, signals, images), which are based on **global and local regularity**

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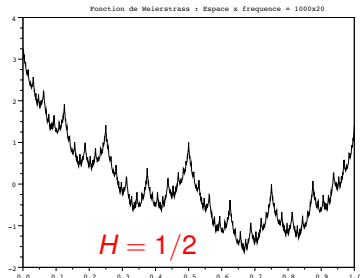
A key problem along the 19th century was to **determine if a continuous function on $\mathbb{R}$ necessarily has points of differentiability**

A first negative answer was obtained by **B. Bolzano** in 1830 but was unnoticed



A second counterexample due to **K. Weierstrass** settled the issue in 1872

Weierstrass functions



$$W_H(x) = \sum_{j=0}^{+\infty} 2^{-Hj} \cos(2^j x)$$

$$0 < H < 1$$

Weierstrass result was sharpened using a continuous scale of pointwise regularity indices

Pointwise regularity

Definition :

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a locally bounded function and $x_0 \in \mathbb{R}^d$;
 $f \in C^\alpha(x_0)$ if there exist $C > 0$ and a polynomial P such that, for
 $|x - x_0|$ small enough,

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha$$

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The Hölder exponent of f at x_0 is

$$h_f(x_0) = \sup\{\alpha : f \in C^\alpha(x_0)\}$$

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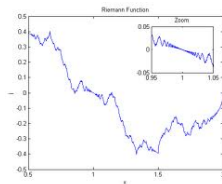
Theorem : (Hardy, 1916)

The Hölder exponent of W_H is constant and equal to H
(W_H is a mono-Hölder function)

Pointwise regularity : Example and open problems

Riemann's nondifferentiable function :

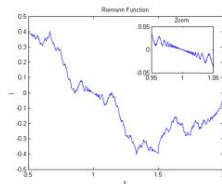
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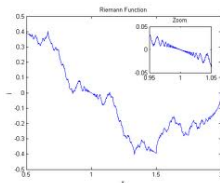


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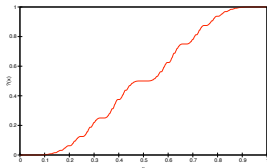
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Minkowski's “question mark” function :

$$?(x) : [0, 1] \rightarrow [0, 1]$$

If $x = [0; a_1, \dots, a_n, \dots]$ then

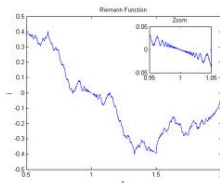
$$?(x) = 2 \sum \frac{(-1)^{n+1}}{2^{a_1 + \dots + a_n}}$$



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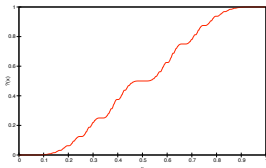
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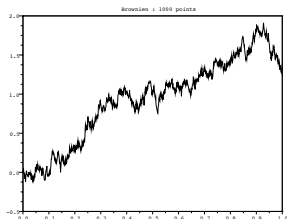


C. Hermite : “Je me détourne avec effroi et horreur de cette plaie lamentable des fonctions qui n'ont pas de dérivée”

H. Poincaré called such functions “monsters”

Brownian motion

Economists (L. Bachelier) and physicists (A. Einstein) put into light the central role played by Brownian motion in modeling

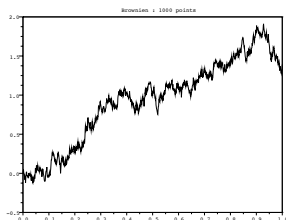


Definition :

Brownian motion is the
unique continuous process
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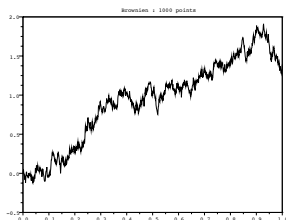
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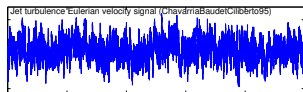
In 1931, S. Banach proved that a “generic” continuous function on $\mathbb{R}$ is nowhere differentiable (in the sense of Baire categories)

Nowhere differentiable functions

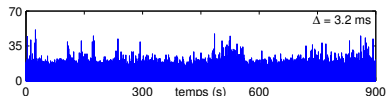
Starting with the example of the surface of colloïds, and the coast of Brittany, J. Perrin, in his book, “Les atomes” published in 1913, insists that such examples, far from being exceptional, supply the right models for natural phenomena

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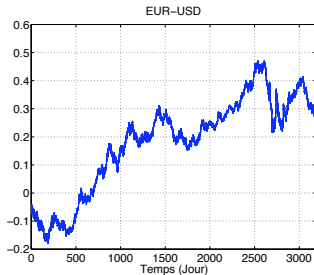
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Fully developed turbulence

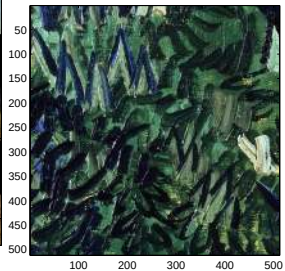
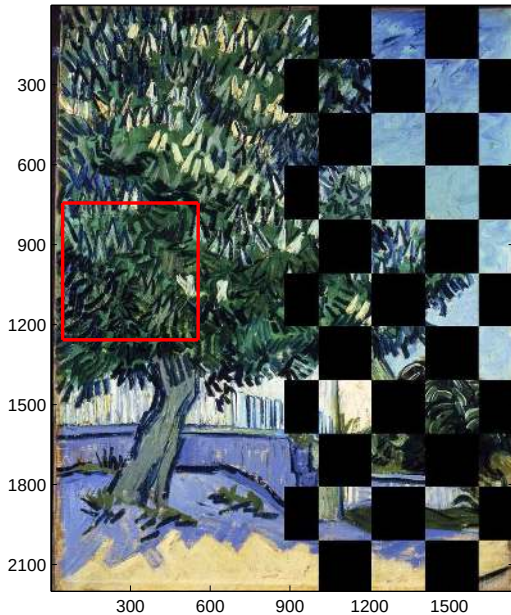


Internet Traffic



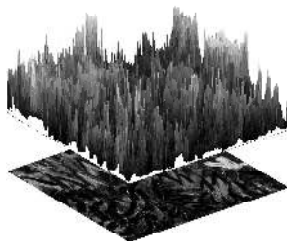
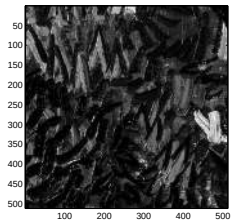
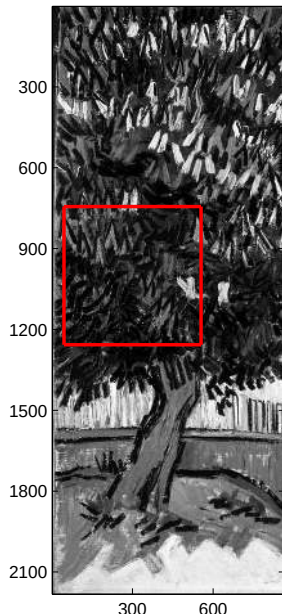
Euro vs Dollar (2001-2009)





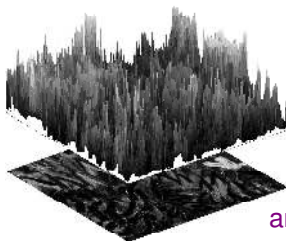
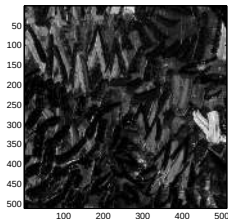
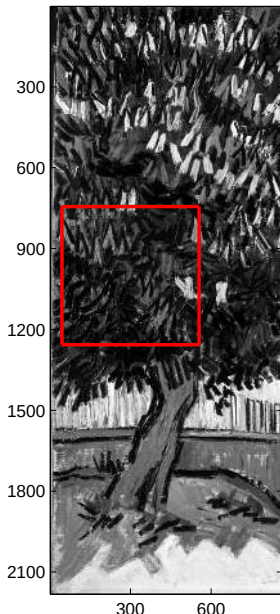
Red channel

f752



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f752



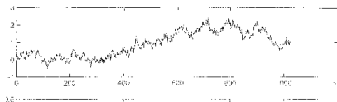
Nature,
Sciences,
and Arts
supply a large variety
of everywhere irregular
functions

Challenge :
Measure this irregularity
and use it for classification
and model selection

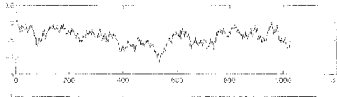
Fractional Brownian Motions

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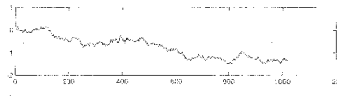
$H = 0.3$



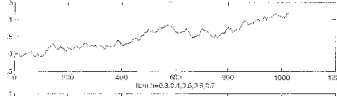
$H = 0.4$



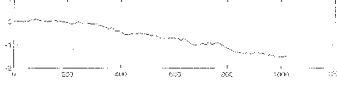
$H = 0.5$



$H = 0.6$



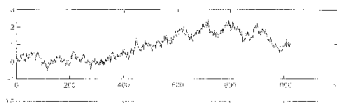
$H = 0.7$



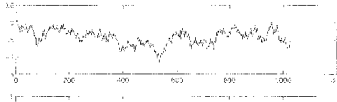
Gaussian
processes
with
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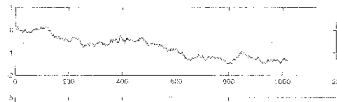
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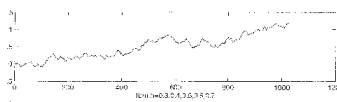
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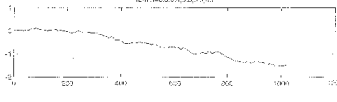
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$H = 0.6$



$H = 0.7$



Gaussian
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Theorem : (A.N. Kolmogorov)

The Hölder exponent of B_H is constant and equal to H

Challenge : Find a numerically stable way to decide if a real-life signal can be modeled by FBM

Kolmogorov scaling function (1941)

$$\int |f(x + \delta) - f(x)|^p dx \sim |\delta|^{\zeta_f(p)}$$

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Numerically : Regression on a log-log plot

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Numerically : Regression on a log-log plot

What is the scaling function of FBM ?

B_H is the unique centered Gaussian process such that

$$\forall x, \delta \geq 0, \quad \mathbb{E}(|B_H(x + \delta) - B_H(x)|^2) = |\delta|^{2H}$$

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$\implies$ Turbulence at small scale cannot be modeled by FBM (1950s)

Functional interpretation : Lipschitz spaces

$$\int |f(x + \delta) - f(x)|^p dx \sim |\delta|^{\zeta_f(p)}$$

Definition : Let $p \in [1, \infty)$; $f \in Lip(s, L^p(\mathbb{R}^d))$ if

$$\exists C > 0, \forall \delta > 0, \quad \|f(\cdot + \delta) - f(\cdot)\|_p \leq C \cdot |\delta|^s$$

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$$\forall p \geq 1, \quad \zeta_f(p) = p \cdot \sup \{s : f \in Lip(s, L^p)\}$$

One can replace the spaces $Lip(s, L^p(\mathbb{R}^d))$ by Sobolev spaces

$$L^{p,s} = \{f \in L^p : (-\Delta)^{s/2} f \in L^p\}$$

The scaling function yields a regularity index in the L^p norm

One-variable wavelet basis

A **wavelet basis** on $\mathbb{R}$ is generated by a smooth, well localized, oscillating function ψ such that the

$\psi(2^j x - k), \quad j, k \in \mathbb{Z}$
form an orthogonal basis of $L^2(\mathbb{R})$

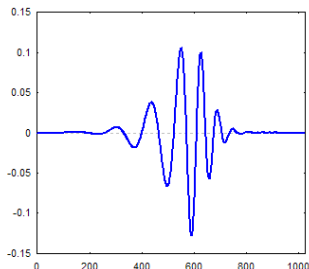
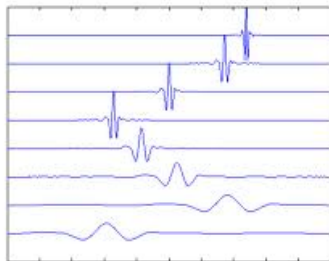
$$\forall f \in L^2(\mathbb{R}),$$

$$f(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} c_{j,k} \psi(2^j x - k)$$

where

$$c_{j,k} = 2^j \int f(x) \psi(2^j x - k) dx$$

Daubechies Wavelet



Notations for wavelets on $\mathbb{R}$

Dyadic intervals

$$\lambda = \left[\frac{k}{2^j}, \frac{k+1}{2^j} \right)$$

Wavelets

$$\psi_\lambda(x) = \psi(2^j x - k)$$

Wavelet coefficients

$$c_\lambda = 2^j \int_{\mathbb{R}} f(x) \psi(2^j x - k) dx$$

Dyadic intervals at scale j

$$\Lambda_j = \{\lambda : |\lambda| = 2^{-j}\}$$

Wavelet expansion of f

$$f(x) = \sum_j \sum_{\lambda \in \Lambda_j} c_\lambda \psi_\lambda(x)$$

Wavelets in 2 variables

In 2D, the wavelets used are tensor products :

$$\psi^1(x, y) = \psi(x)\varphi(y)$$

$$\psi^2(x, y) = \varphi(x)\psi(y)$$

$$\psi^3(x, y) = \psi(x)\psi(y)$$

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Notations

Dyadic squares :

$$\lambda = \left[\frac{k}{2^j}, \frac{(k+1)}{2^j} \right] \times \left[\frac{l}{2^j}, \frac{(l+1)}{2^j} \right]$$

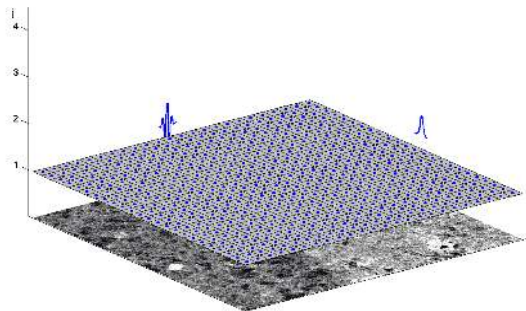
Wavelet coefficients

$$c_\lambda = 2^{2j} \int \int f(x, y) \psi^i \left(2^j x - k, 2^j y - l \right) dx dy$$

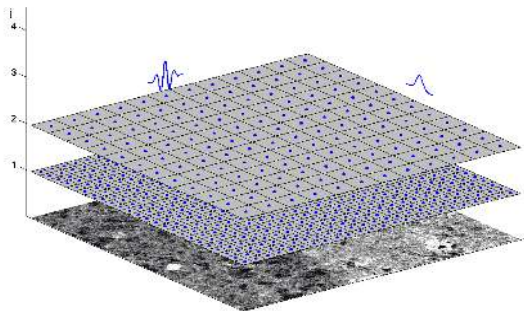
Computation of 2D wavelet coefficients



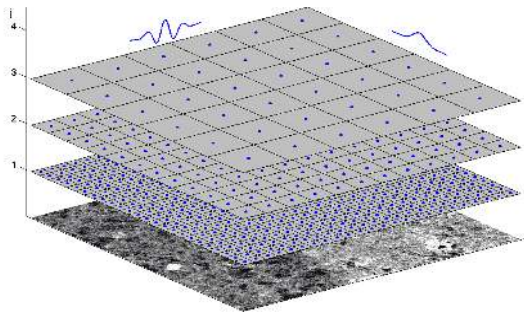
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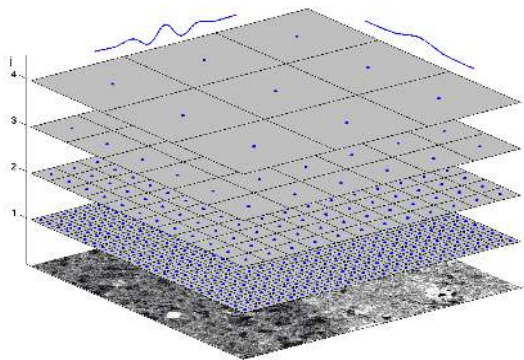
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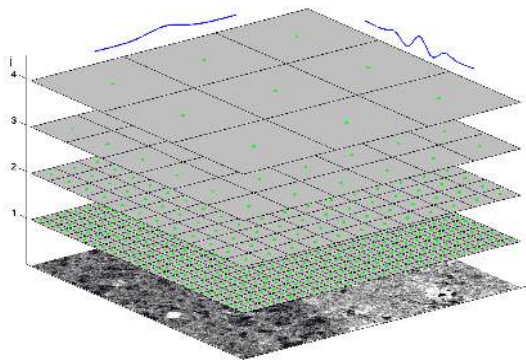
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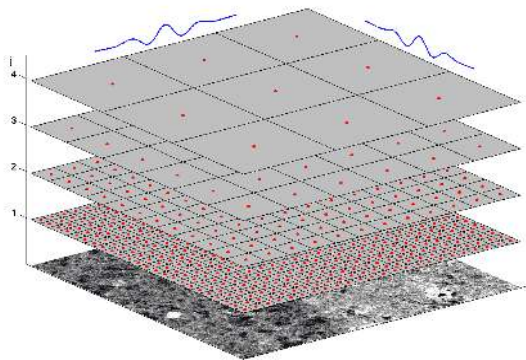
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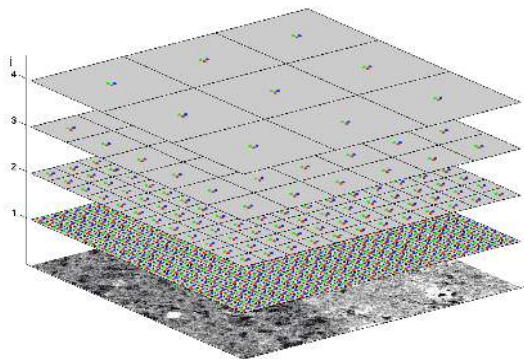
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The wavelet scaling function

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$; its **wavelet scaling function** is defined $\forall p > 0$ by

$$2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \sim 2^{-\zeta_f(p)j} \quad \text{i.e.} \quad \zeta_f(p) = \liminf_{j \rightarrow +\infty} \frac{\log \left(2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \right)}{\log(2^{-j})}$$

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Besov spaces : Let $p > 0$; $f \in B_p^{s,\infty}(\mathbb{R}^d)$ if

$$\exists C, \forall j : \quad 2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \leq C \cdot 2^{-spj}$$

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Embeddings between Lipschitz and Besov spaces imply that, **when $p \geq 1$** , the wavelet scaling function coincides with Kolmogorov's scaling function

The role of the wavelet scaling function

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- ▶ If $\zeta_f(1) > 1$, then $f \in BV$
- ▶ If f is a measure, then $\zeta_f(1) \geq 0$
- ▶ If $\zeta_f(1) > 0$, f then belongs to L^1
- ▶ If $\zeta_f(2) > 0$, then $f \in L^2$

Motivations :

- ▶ Y. Gousseau, J.-M. Morel : *Are natural images of bounded variation ?* (SIAM J. Math. Anal., Vol. 3, 2001)
- ▶ Jump models and finite quadratic variation assumption in finance

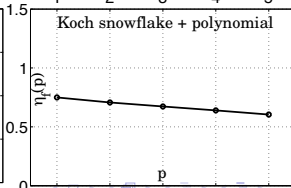
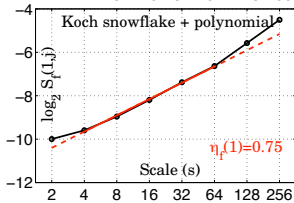
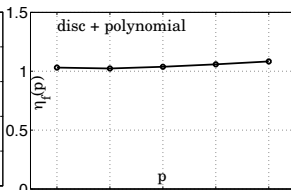
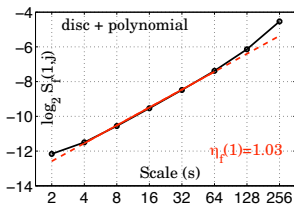
Wavelet scaling functions of synthetic images

Wavelet scaling function $\zeta_f(p)$:

$$2^{-2j} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \sim 2^{-\zeta_f(p) j}$$

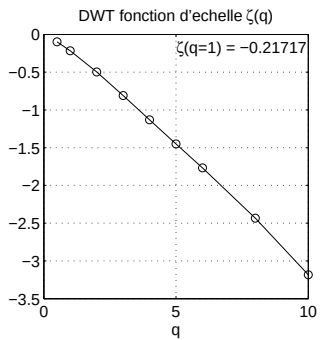
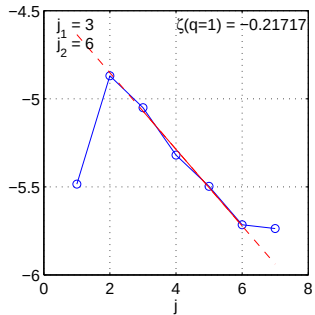
Disk : $\zeta_f(p) = 1$

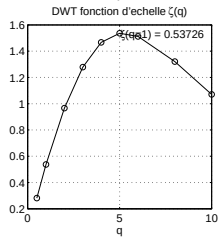
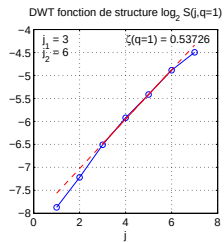
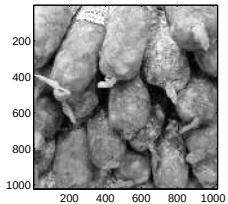
Van Koch snowflake : $\zeta_f(p) = 2 - \frac{\log 4}{\log 3} \sim 0.74$



Natural images







Uniform Hölder regularity

Hölder spaces : Let $\alpha \in (0, 1)$; $f \in C^\alpha(\mathbb{R}^d)$ if

$$\exists C, \forall x, y : |f(x) - f(y)| \leq C \cdot |x - y|^\alpha$$

$$\forall \alpha \in \mathbb{R}, \quad C^\alpha(\mathbb{R}^d) = B_\infty^\alpha(\mathbb{R}^d)$$

The uniform Hölder exponent of f is

$$H_f^{min} = \sup\{\alpha : f \in C^\alpha(\mathbb{R}^d)\}$$

Numerical computation

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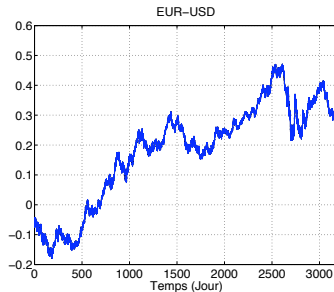
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$$H_f^{\min} > 0 \implies f \text{ is continuous}$$

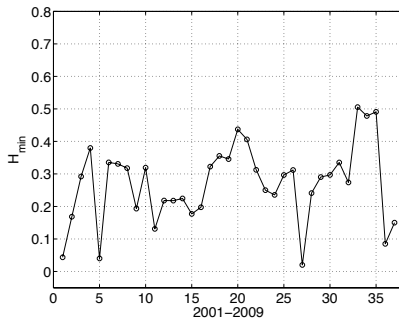
$$H_f^{\min} < 0 \implies f \text{ is not locally bounded}$$

Validity of jump models in finance

Euro vs. USD
2001-2009



H_f^{min}

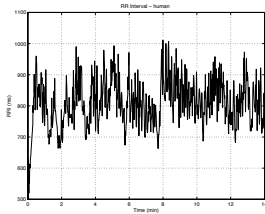


Data supplied by Vivienne Investissement

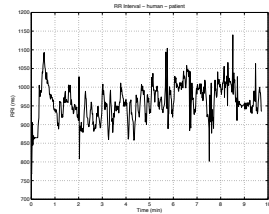
Classification based on the uniform Hölder exponent

Heartbeat intervals

Healthy



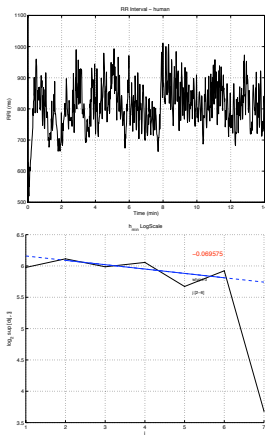
Heartbeat failure



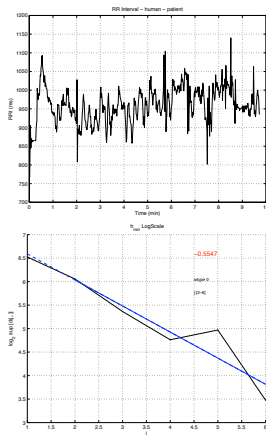
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Function space regularity :

Validity of stochastic integration tools in finance

Definition : A function $f : \mathbb{R} \rightarrow \mathbb{R}$ has **finite quadratic variation** if

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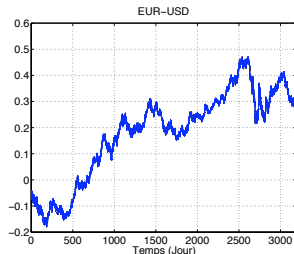
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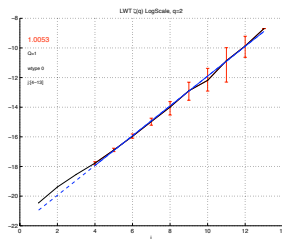
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Euro vs. USD (2001-2009)



$$\zeta_f(2) = 1.0053$$

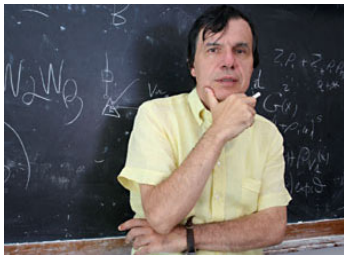
Limitations

Classification only based on the wavelet scaling function or on the uniform Hölder exponent proved insufficient in several occurrences (turbulence, data mining, ...)

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Further developments were based on seminal ideas introduced by U. Frisch and G. Parisi, and paved the way to the construction of a new scaling function



Giorgio Parisi



Uriel Frisch

Multifractal spectrum (Parisi and Frisch, 1985)

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Let f be a locally bounded function. The **multifractal spectrum** of f is

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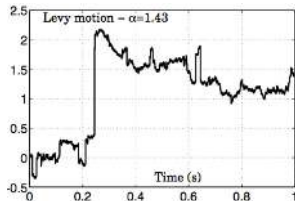
Parisi and Frisch's fundamental idea was that the nonlinearity of the scaling function reflects the presence of a whole range of fractal sets E_H , and that the scaling function yields information on the “sizes” of these sets

Two results showed that multifractal analysis does not only concern “strange examples” :

I : Probabilistic result :

Definition : A Lévy process is a stochastic process with independent and stationary increments, i.e. :

$X_{t+s} - X_t$ is independent of the $\{X_u, u \leq t\}$ and has the same law as X_s

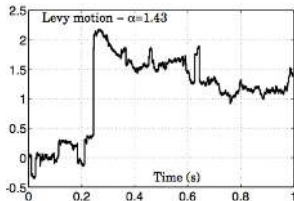


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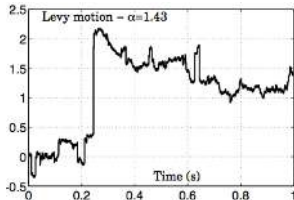
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In each case, the sets E_H are everywhere dense



The numerical determination of the Hölder exponent is hopeless

II : Generic results :

Definition : Let E be a metric Banach space. A Borel set $A \subset E$ is **Haar null** if there exists a compactly supported probability measure μ on E such that

$$\forall x \in E, \quad \mu(A + x) = 0$$

A set is **prevalent** if its complement is Haar null

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Theorem : (A. Fraysse and S. J.) Let $s > d/p$; then quasi-every and almost every function f of $L^{p,s}(\mathbb{R}^d)$ is multifractal, and its spectrum is given by

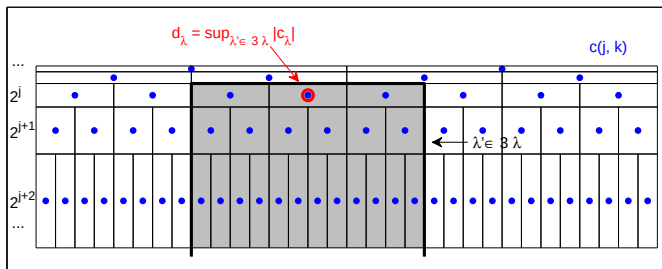
$$D_f(H) = \begin{cases} p(H - s) + d & \text{if } H \in \left[s - \frac{d}{p}, s\right] \\ -\infty & \text{else} \end{cases}$$

Wavelet leaders

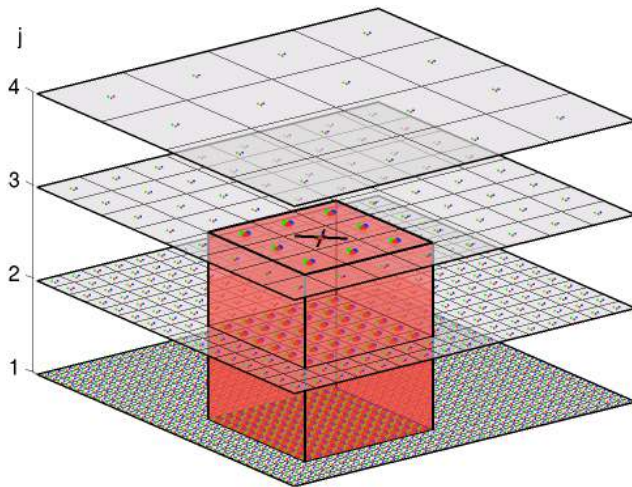
Let λ be a dyadic cube ; 3λ denotes the cube of same center and three times wider

Let f be a **locally bounded function** ; the **wavelet leaders** of f are

$$d_\lambda = \sup_{\lambda' \subset 3\lambda} |c_{\lambda'}|$$



Computation of 2D wavelet leaders



Wavelet leaders allow to estimate the pointwise Hölder exponent

Leader scaling function

Wavelet scaling function

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Advantages :

- ▶ It is “well defined” **for all $p \in \mathbb{R}$**
- ▶ For p large enough, $\zeta_f(p) = \eta_f(p)$
- ▶ If $\psi^i \in \mathcal{S}(\mathbb{R}^d)$, then $\forall p \in \mathbb{R}$, $\eta_f(p)$ is independent of the wavelet basis
- ▶ η_f is invariant under **“smooth perturbations”** of f

Multifractal formalism

Since η_f is a concave function, there is no loss of information in rather considering its Legendre transform :

The **Legendre Spectrum** of f is

$$L_f(H) = \inf_{p \in \mathbb{R}} (d + Hp - \eta_f(p))$$

Theorem : Let $D_f(H)$ denote the Hausdorff dimension of the set of points where $h_f(x) = H$. If $f \in C^\varepsilon(\mathbb{R}^d)$ for an $\varepsilon > 0$ then

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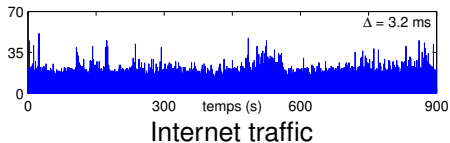
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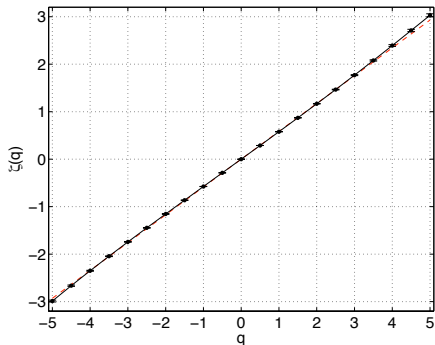
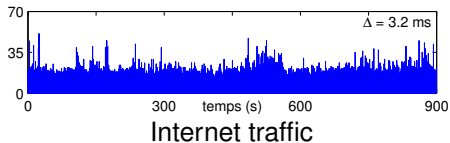
The multifractal formalism is satisfied when equality holds

Open problem : Find “reasonably weak” general hypotheses implying the validity of the multifractal formalism

Monohölder vs. Multifractality



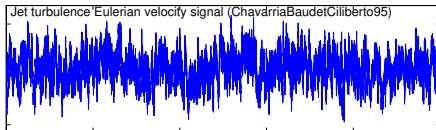
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Wavelet leader scaling function

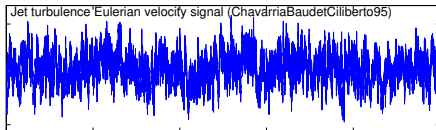
Model refutation : Fully developed turbulence

(joint work with Bruno Lashermes)

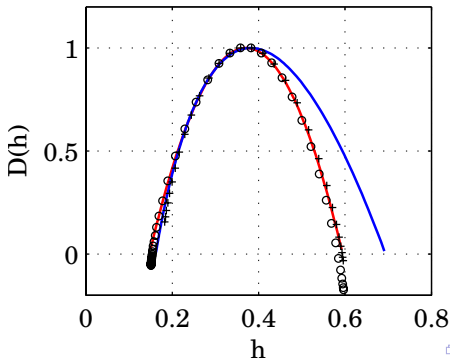


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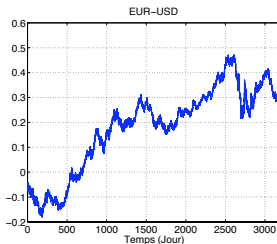


Log-normal vs. Log-Poisson model



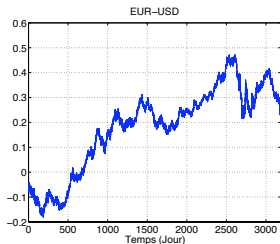
Evidence of time-evolution : Finance

Multifractal analysis
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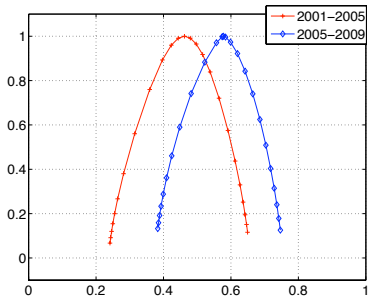


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Legendre
spectra

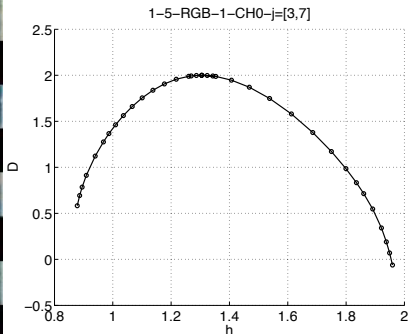


Multifractal analysis of paintings : Van Gogh challenge

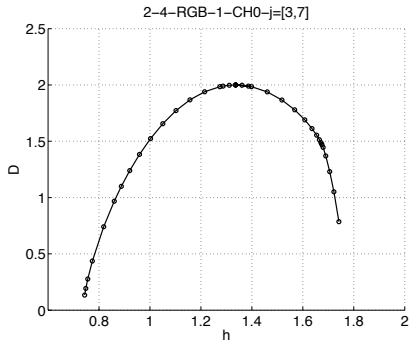
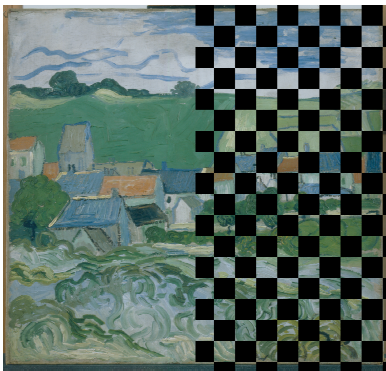
Initiated by the Van Gogh Museum (Amsterdam)

Coordinated by I. Daubechies and R. Johnson

(joint work with D. Rockmore)



Van Gogh : Arles and Saint-Rémy period



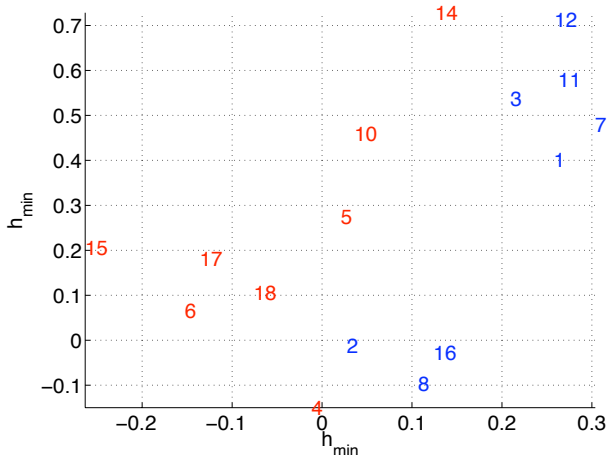
Van Gogh : Paris period

Challenge : Date

Paris period - Arles, Saint-Rémy period - Unknown

Canals : Red vs. Saturation

$h_{\min} - h_{\min}; j=[1,4]/[1,4] - \text{RGB } 0/1 - \text{CH } 2/1$

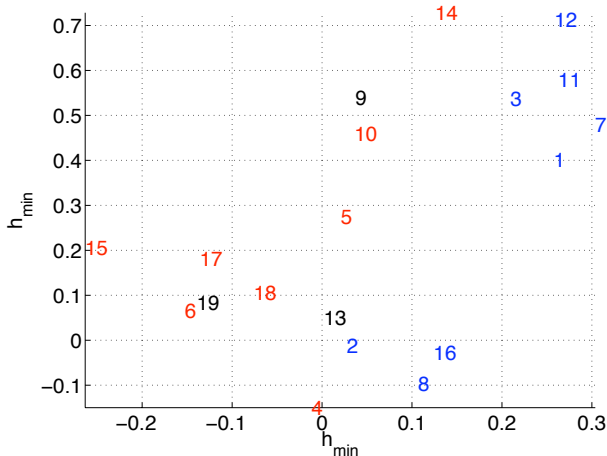


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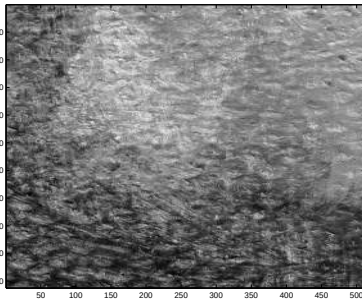
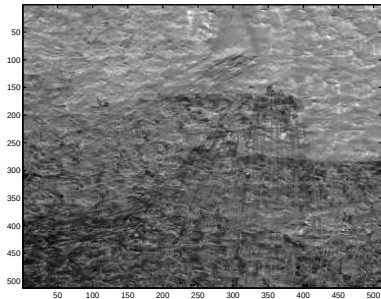
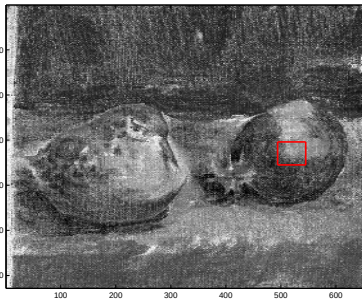
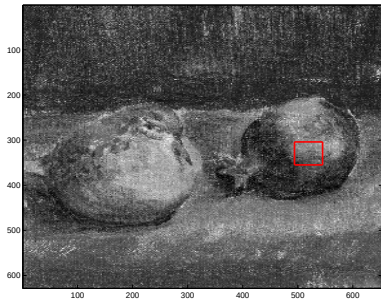
Original and copy : Stylometry issues

Experiment initiated by I. Daubechies

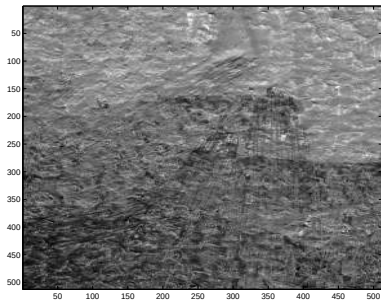


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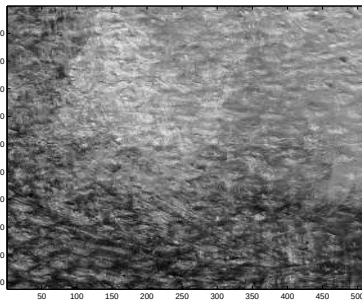
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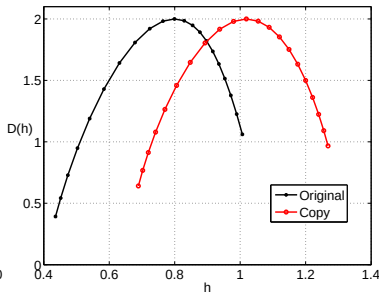
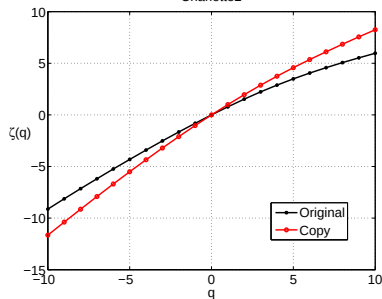
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